$\mathbb{A}^n$ vs. $\mathbb{P}^n$

Question: How do algebraic sets (/ fields of rat'l functions / local rings) in $U_{n+1} = \mathbb{A}^n$ relate to those in the $\mathbb{P}^n$ it sits inside?

$\mathbb{A}^n$ to $\mathbb{P}^n$: Let $V \subseteq \mathbb{A}^n$ be an algebraic set w/
$I = I(V) \subseteq k[x_1, \ldots, x_n]$.

Let $J \subseteq k[x_1, \ldots, x_{n+1}]$ be the ideal generated by the homogenization of elements of I (check: J is still radical)

Thus, J is homogeneous, and define $\overline{V} := V_p(J) \subseteq \mathbb{P}^n$

$\overline{V}$ is called the <u>projective closure</u> of V in $\mathbb{P}^n$.
(In fact, it's the Zariski closure of V in $\mathbb{P}^n$).
see HW

Check: $\overline{V} \cap U_{n+1} = V$

Caution: In general, it is <u>not</u> true that if $I = (f_1, \ldots, f_r)$ then $J = (F_1, \ldots, F_r)$ where F_i = homog. of f_i.

Ex: $I = (x^2+y, x)$. Then $y \in I$, so $y \in J$ = homog. of I, but $y \notin (x^2+yz, x) = J'$.

Geometrically: $V_a(I) = \{(0,0)\}$, $V_p(J) = \{[0:0:1]\}$

but $V_p(J') = \{[0:0:1],[0:1:0]\}$, i.e. we get an extra point in this case if we just homogenize generators.

Ex: Recall $I = (x^2 - y,\ z - xy) \subseteq \mathbb{C}[x,y,z]$. We showed on a past HW that $V_a(I) = \{(t, t^2, t^3)\} \subseteq \mathbb{A}^3$.

This is called the <u>twisted</u> <u>cubic</u>.

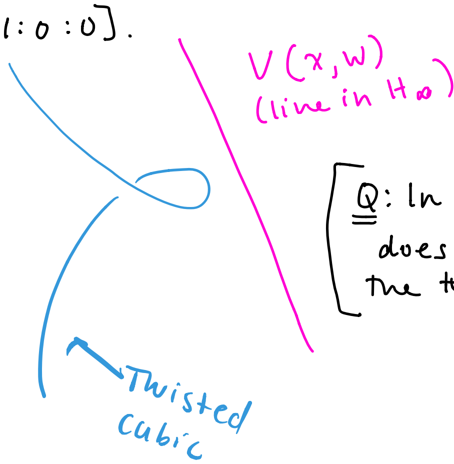
Let J be the homogenization of I in $k[x,y,z,w]$.

Define J' to be $J' = (x^2 - yw,\ zw - xy)$

Clearly, $V_p(J') \cap U_4 = V_a(I)$.

<u>But</u> if $w=0$ in $V_p(J')$, then $x=0$, and y and z can be arbitrary, so

$V_p(J') = V_p(J) \cup V_p(x,w)$, which is reducible, since $[0:1:0:0] \in V_p(x,y)$ but $zx - y^2 \in I \Rightarrow zx - y^2 \in J$ which doesn't vanish at $[0:1:0:0]$.



[Q: In how many points does $V(x,w)$ intersect the twisted cubic?]

(Interesting note: We <u>can't</u> actually define the twisted cubic in $\mathbb{P}^3$ using only two equations. That is, the twisted cubic is not what's called a <u>complete intersection</u>. i.e. it can't be defined by # of equations equal to its codimension. One way to see this is that it has degree 3, whereas the intersection of two quadrics (deg 2 hypersurfaces) has degree 4 – hence the additional line. Thus, you need an additional quadric to cut out the twisted cubic. – See HW)

$\mathbb{P}^n$ to $\mathbb{A}^n$

$V \subseteq \mathbb{P}^n$ an algebraic set, $I = I_p(V) \subseteq k[x_1, \dots, x_{n+1}]$

Then intersecting with U_{n+1} corresponds to dehomogenizing, i.e. taking $\underset{\overline{I}}{\underset{\shortparallel}{I/(x_{n+1}-1)}} \subseteq k[x_1, \dots, x_{n+1}]/(x_{n+1}-1) \cong k[x_1, \dots, x_{n+1}]$

Note: If we think of I as the ideal of the cone $C(V)$, then the quotient $k[x_1, \dots, x_{n+1}] \to k[x_1, \dots, x_{n+1}]/(x_{n+1}-1)$ corresponds to the inclusion $\underset{V_a(x_{n+1}-1)}{\underset{\shortparallel}{U_{n+1}}} \hookrightarrow \mathbb{A}^{n+1}$

Then $\overline{I}$ is the image of I in the quotient, so $V(\overline{I}) = C(V) \cap U_{n+1}$
$= V \cap U_{n+1}$

Fields of rational functions

Let $V \subseteq \underset{U_{n+1}}{\overset{\shortparallel}{\mathbb{A}^n}}$ be an affine variety and $W \subseteq \mathbb{P}^n$ its projective closure. We can define a map

$$\alpha : \underset{\substack{\uparrow \\ \text{quotients} \\ \text{of forms of} \\ \text{the same deg}}}{k(W)} \longrightarrow \underset{\substack{\uparrow \\ \text{field of} \\ \text{fractions of} \\ \Gamma(V)}}{k(V)} \quad \text{by} \quad \alpha\left(\frac{F}{G}\right) = \frac{F(x_1, \dots, x_n, 1)}{G(x_1, \dots, x_n, 1)}$$

(subtlety: can you see why this is well-defined?)

Prop: α is an isomorphism.

Pf: Let $\frac{a}{b} \in k(V)$. Let A and B be the homog. of a, b resp. and $\underset{\substack{\uparrow \\ \text{maybe } < 0}}{q} = \deg b - \deg a$. Then $\frac{A}{B} x_{n+1}^q \longmapsto \frac{a}{b}$. Thus, α is a surjection of fields and thus an isomorphism. $\square$

Cor: If $P \in V$, α induces an isom $\mathcal{O}_P(W) \to \mathcal{O}_P(V)$.

Pf: $\alpha(\mathcal{O}_P(W)) \subseteq \mathcal{O}_P(V)$ since F defined at $P \Rightarrow F(x_1, \dots, x_n, 1)$ is.

The map is injective since α is.

If $\frac{a}{b} \in \mathcal{O}_P(V)$, then the homogenization of b will be nonzero at P as well, so the map is surjective. $\square$

Remark: If $\overset{\text{we choose}}{P \notin U_{n+1}}$, we can choose a different U_i s.t. $P \in U_i$, and dehomogenize w.r.t. x_i.